Breather compactons in nonlinear Klein-Gordon systems

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We demonstrate the existence of a localized breathing mode with a compact support, i.e., a stationary breather compacton, in a nonlinear Klein-Gordon system. This breather compacton results from a delicate balance between the harmonicity of the substrate potential and the total nonlinearity induced by the substrate potential and the coupling forces between adjacent lattice sites. [S1063-651X(99)00811-9]

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The soliton concept appeared for the first time in the context of nonlinear lattices, before becoming a reality in many branches of science [1]. In particular, the long-range interaction of solitons is a crucial problem which has been intensively investigated for both its fundamental and applied interests. For example, in nonlinear fiber optics, this long-range interaction imposes a strict limitation on the performance of long-haul fiber transmissions [2]. The concept of compactification or strict localization of solitary waves appeared recently in the literature. Rosenau and Hyman [3], who investigated a special type of Korteweg–de Vries equation, discovered that solitary waves may compactify in the presence of a nonlinear dispersion. Such solitary waves, which are characterized by a compact support, i.e., the absence of infinite wings, have been called *compactons*. In other words, two adjacent compactons do not interact unless they come into contact in a way similar to the contact between hard spheres.

Dusuel *et al.* [4] demonstrated the existence of a static compacton in a real physical system, made up of identical pendulums connected by springs. Moreover, traveling compactonlike kinks may exist in nonlinear Kein-Gordon lattices with Φ-four potential, in conditions that require the presence of nonlinear dispersion and the absence of linear dispersion [5]. Thus, an understanding of the physical mechanisms which give rise to compactons is essential for predicting the conditions in which real physical systems can support such compact structures. The possibility of the existence of breather compactons was predicted in recent studies [6,7]. Kivshar [6] reported that breathers with a compact support may exist in a lattice of identical particles interacting via purely anharmonic coupling forces, without any on-site substrate potential. These results raise a fundamental question: can a compactonlike breather survive the effects of an on-site substrate potential? The answer to this question is given in the present work.

In this Brief Report, we demonstrate the existence of a breather with a compact support, i.e., a breather compacton (BC), in a nonlinear Klein-Gordon lattice with a soft on-site substrate potential. We obtain the exact analytical compacton solution in the continuum lattice, which agrees surprisingly well with the exact numerical solution, not only in the continuum system but also in the discrete and highly discrete system. Using a deformable substrate potential with a parameter that measures the importance of harmonicity in the substrate potential, we show that a BC results from a balance between the harmonicity of the substrate potential and the total nonlinearity induced by the substrate potential and the coupling forces between adjacent lattice sites. In other words, we show that the compactification of a nonlinear Klein-Gordon breather requires the presence of harmonicity in the substrate potential. Our study reveals the parameter regions in which a BC can execute a stable oscillatory motion.

The system under consideration is a nonlinear Klein-Gordon system with nonlinear coupling between lattice sites, governed by the following Hamiltonian:

$$H=\sum_n \tfrac{1}{2}\dot{Q}_n^2+\tfrac{1}{4}C_{\rm nl}(Q_n-Q_{n-1})^4+\tfrac{1}{8}\omega_0^2V(Q_n), \qquad (1)$$

where Q_n is the position of the nth particle measured from the nth lattice site, $C_{\rm nl}$ is that parameter that controls the strength of the nonlinear coupling, the dot indicates the time derivative, ω_0 is the limiting frequency for long-wavelength excitations, and $V(Q_n)\equiv\alpha Q_n^2/2+Q_n^4/4$ is the soft on-site potential. The parameter α, which serves as a measure of the importance of harmonicity in the substrate potential, plays a crucial role for stabilizing a BC, as we show later on. It is worth noting in Eq. (1) the absence of linear coupling between adjacent lattice sites. The reason is that the linear coupling forces would give rise to a phonon band which may enter in direct resonance with the internal modes of a compacton, thus causing radiation of energy away from the compacton. The lattice equations of motion are

$$\frac{\partial^2 Q_n}{\partial t^2}=C_{\rm nl}[(Q_{n+1}-Q_n)^3+(Q_{n-1}-Q_n)^3] -\tfrac{1}{2}\omega_0^2(\alpha Q_n+Q_n^3). \qquad (2)$$

For $\omega_0^2\ll C_{\rm nl}$, one approaches the continuum limit, in which Q_n varies slowly from one site to another. Thus, using the continuum limit approximation, Eq. (2) can be reduced to the following partial differential equation [4]:

$$\frac{\partial^2 Q}{\partial t^2}-3C_{\rm nl}\left(\frac{\partial Q}{\partial x}\right)^2\frac{\partial^2 Q}{\partial x^2}+\tfrac{1}{2}\omega_0^2(\alpha Q+Q^3)=0, \qquad (3)$$

where the site index n is replaced by the continuous position variable x.

To obtain the stationary breather solution, we look for localized waves of the form $Q(x,t)=\theta(t)\phi(x)$, which we substitute in Eq. (3) to obtain the following equations:

$$\theta_{tt}+\tfrac{1}{2}\alpha\tfrac{1}{2}\omega_0^2\theta+K^2\theta^3=0, \tag{4a}$$

$$3C_{\rm nl}\phi_x^2\phi_{xx}-\tfrac{1}{2}\omega_0^2\phi^3+K^2\phi=0, \tag{4b}$$

where K is an arbitrary constant. Solving Eqs. (4), we obtain, after a little algebra, the following continuum solution for breathers with compact support (or compactonlike breather):

$$Q(x,t)=Q_M\cos(\beta x)cn[(\alpha+Q_M^2)^{1/2}\omega_0 t/\sqrt{2},k^2]$$

$$\text{for } |x-X|\le\alpha=\pi/(2\beta), \tag{5}$$

$$Q(x,t)=0 \quad \text{for } |x-X|>\alpha, \quad k\equiv Q_M/[2(\alpha+Q_M^2)]^{1/2},$$

where X locates the center of mass of the breather. The parameter $\beta\equiv[\omega_0^2/(6C_{\rm nl})]^{1/4}$ may serve as a measure of the importance of discreteness effects in the system. The solution given by Eq. (5) indicates that a BC possesses a cosine spatial profile. Its amplitude is proportional to Q_M, which multiplies a *cn* Jacobi elliptic function of time. Whereas a standard breather soliton possesses exponential (infinite) wings, the full width of the BC is strictly limited to

$$L_C=\pi/\beta=\pi[6C_{\rm nl}/\omega_0^2]^{1/4}, \tag{6}$$

thus implying that two BC's will not interact unless they come into contact in a way similar to the contact between two hard spheres.

To check the validity of the above compacton solution (5), we have performed the numerical solution of the equations of motion (2), which treat intrinsically the lattice discreteness (i.e., in which no continuum limit approximation is made). We choose the initial condition for the simulation to be $Q_n(t=0)=Q_M\cos(\beta n)$ for the particles which make up the breather motion. This breather profile corresponds to the solution (5) evaluated at discrete lattice sites. The initial velocities on the particles are chosen to be zero. Hereafter we take $\omega_0^2=8$. Figure 1 shows the evolution of the lattice profile over one period of the breather motion. The largest curves correspond to a situation where the full width of the compacton is $L_C=57$ times the lattice spacing [or equivalently $C_{\rm nl}=\frac{4}{3}(57/\pi)^4$], which therefore corresponds to a situation close to the continuum system. As can be seen in Fig. 1, the analytical continuum solution (5) represented by the largest solid curves agrees extremely well with the exact solution that we obtained numerically (dotted curve). After one period of the dynamics ($T=0.18$), the breather recovers the exact shape it has initially at $t=0$.

On the other hand, the smallest curves in Fig. 1 show the results that we obtained for $L_C=6$, which corresponds to a quite discrete system. Unexpectedly, we find there that the continuum solution (5) agrees surprisingly well with the exact discrete solution (dotted curves). Thus, the results in Fig. 1 indicate that discreteness effects may not be detrimental to the abrupt localization of nonlinear Klein-Gordon breathers. This feature is remarkably well confirmed by the results represented in Fig. 2, which we obtained for $L_C=2$. This situation in which only a single particle makes up the breather motion corresponds to the ultimate degree of discretization of a breather mode. Consequently, it would be inappropriate defining a precise spatial profile for such a discrete breather. However, Fig. 2 shows, contrary to common intuition, that the particle that makes up the breather motion is perfectly described by the continuum solution (5) evaluated at $x=4$, the center of mass of the breather. *Nevertheless, from the above results, we cannot conclude that here we have a breather compacton, but rather a discrete breather with abrupt localization.*

Whereas the *abrupt localization* or quasicompactification of a nonlinear Klein-Gordon breather seems to be relatively insensitive to the importance of the effects of lattice discreteness, the presence of harmonicity in the substrate potential is a major requirement for the existence of a BC. Indeed we have found out that the harmonicity parameter α determines the maximum amplitude of the breather motion, $Q_{\rm max}$, below which the breather is stable ($0<Q_M\le Q_{\rm max}$). The higher the

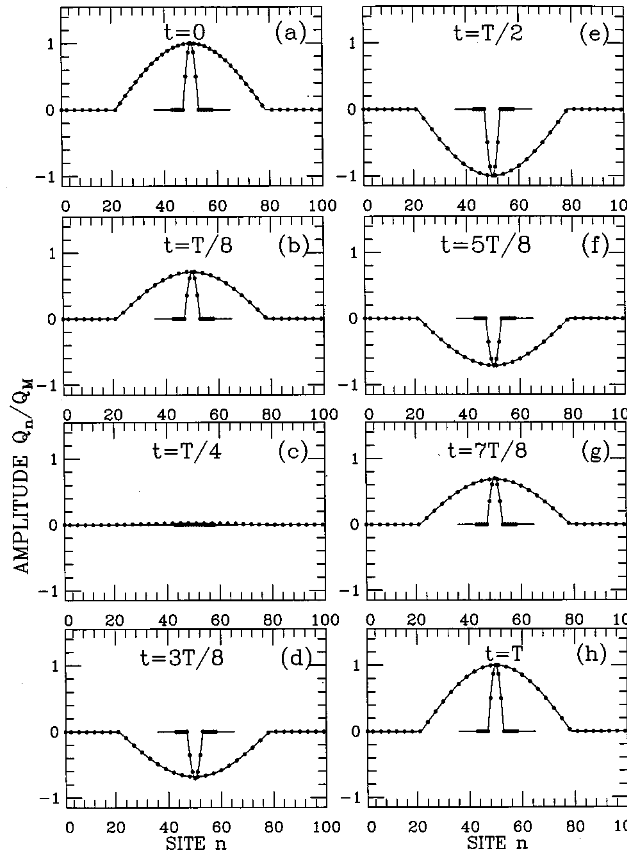

FIG. 1. Plots showing the temporal evolution of the spatial profile of a compactonlike breather over one period of the dynamics: $T=0.181$. The largest curves correspond to $L_C=57$ [$C_{\rm nl}=\frac{4}{3}(57/\pi)^4$]. The smallest curves correspond to $L_C=6$ [$C_{\rm nl}=1728/\pi^4$]. The solid curves show the analytical solution (5) in the continuum approximation. The dotted curves represent the exact solution of the discrete equations of motion (2).

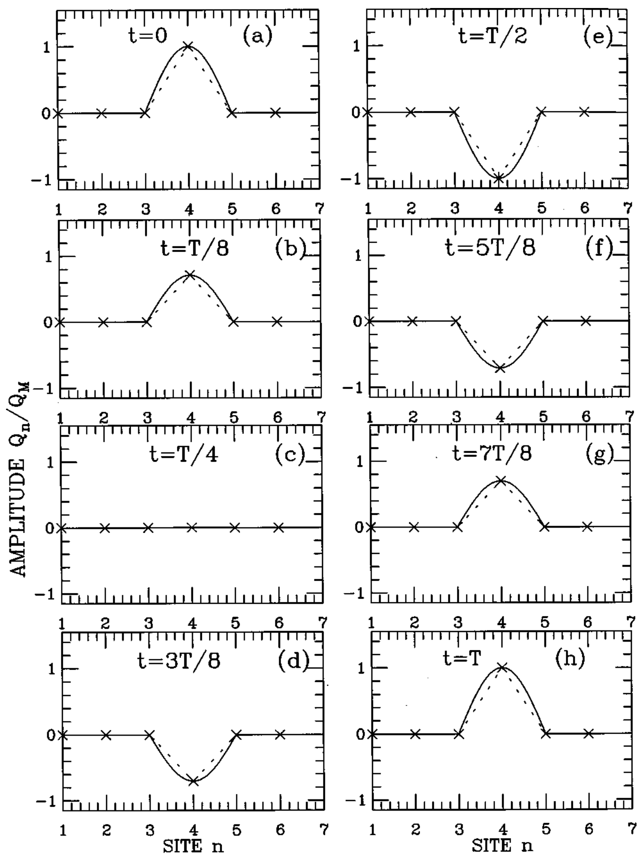

FIG. 2. Representation of the temporal evolution of the breather over one period of the dynamics in the ultimate degree of discretization ($T=0.181$). The simulation parameters are $L_C=2$ [$C_{\mathrm{nl}}=\frac{4}{3}(57/\pi)^4$]. The solid curves show the analytical continuum solution (5). The crosses represent the exact solution of the discrete equations of motion (2).

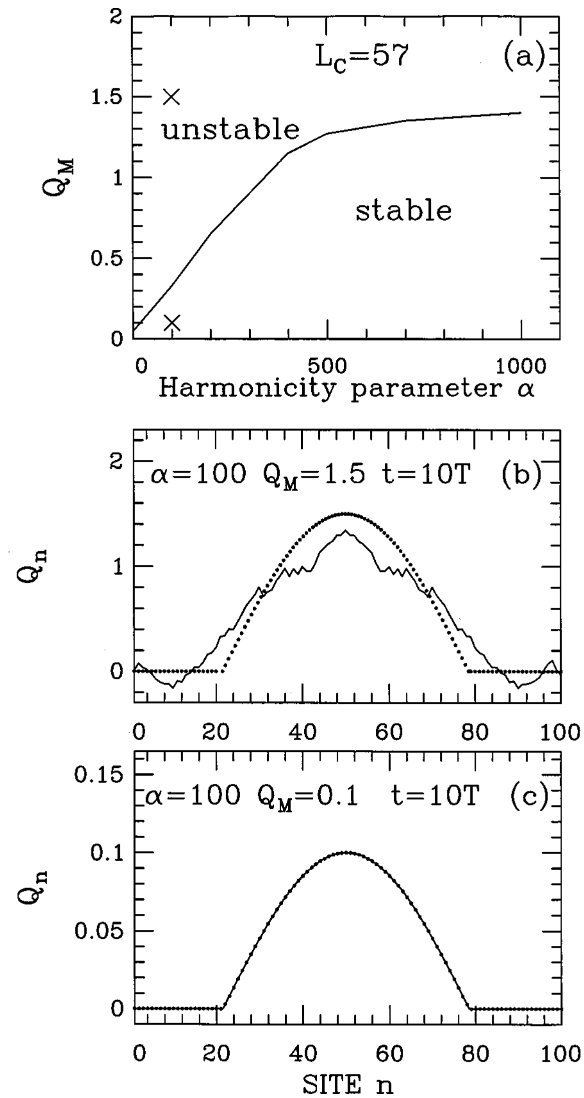

FIG. 3. Plot showing the parameter region of stability of the compacton for $L_C=57$, in the range $1\leqslant\alpha\leqslant 1000$. (a) Stability diagram. (b) Breather profile for $Q_M=1.5$. (c) Breather profile for $Q_M=0.1$. The dotted curves show the breather profile at $t=0$. The solid curves show the breather profile at $t=10T\approx 3.14$.

harmonicity parameter, the higher the maximum amplitude Q_{max}. Figure 3(a) shows the stability diagram of a BC in the range $1\leqslant\alpha\leqslant 1000$. Figures 3(b) and 3(c), which we obtained for $\alpha=300$, show the breather profile after ten periods of its oscillatory motion ($t=10T\approx 3.14$), for unstable and a stable points of the stability diagram, respectively. Those two points are indicated by the two crosses in Fig. 3(a). Figure 3(b) shows that in the parameter region of instability, the spatial profile of the breather and its temporal periodicity are destroyed. Quite in contrast, in the parameter region of stability the breather motion becomes perfectly periodic and the cosine shape of its spatial profile is preserved during the motion. Furthermore, we have performed simulations not represented here, which show that in general the domain of stability of a BC increases as the nonlinear coupling decreases, that is, as the system is more and more discrete.

In conclusion, we have demonstrated the existence of localized breather modes with compact support in a standard nonlinear Klein-Gordon system. Whereas the compactification of nonlinear Klein-Gordon kinks requires a strong nonlinear coupling [4,5], that is, a condition close to the continuum limit, we have shown that contrary to common intuition the quasicompactification or abrupt localization of a nonlinear Klein-Gordon breather can be achieved in discrete systems. Furthermore, whereas the harmonic coupling must be absent to avoid any radiation of energy away from a compacton [5], it comes out from this research that the presence of harmonicity in the substrate potential is required to stabilize a BC, a result which remains to be explained. This harmonicity gives rise to a phonon line, which corresponds to standing waves that cannot propagate owing to their zero group velocity. Finally, it emerges from this Brief Report as well as from previous related studies [4–6] that the concept of compactification of solitary waves implies strict conditions that must be satisfied in order for physical systems to support localized modes with compact support. We have pointed out some of those conditions in the present work. Nevertheless, in the actual stage of the research on structures with compact support, the results that have been obtained are still far away from practical applications. Much work remains to be done, and in particular some fundamental problems remain to be carefully examined, such as the interaction of such structures. This effort deserves to be carried out to make the compacton concept a reality in some areas in which compactons could ensure practical applications such as in signal processing and communications.

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